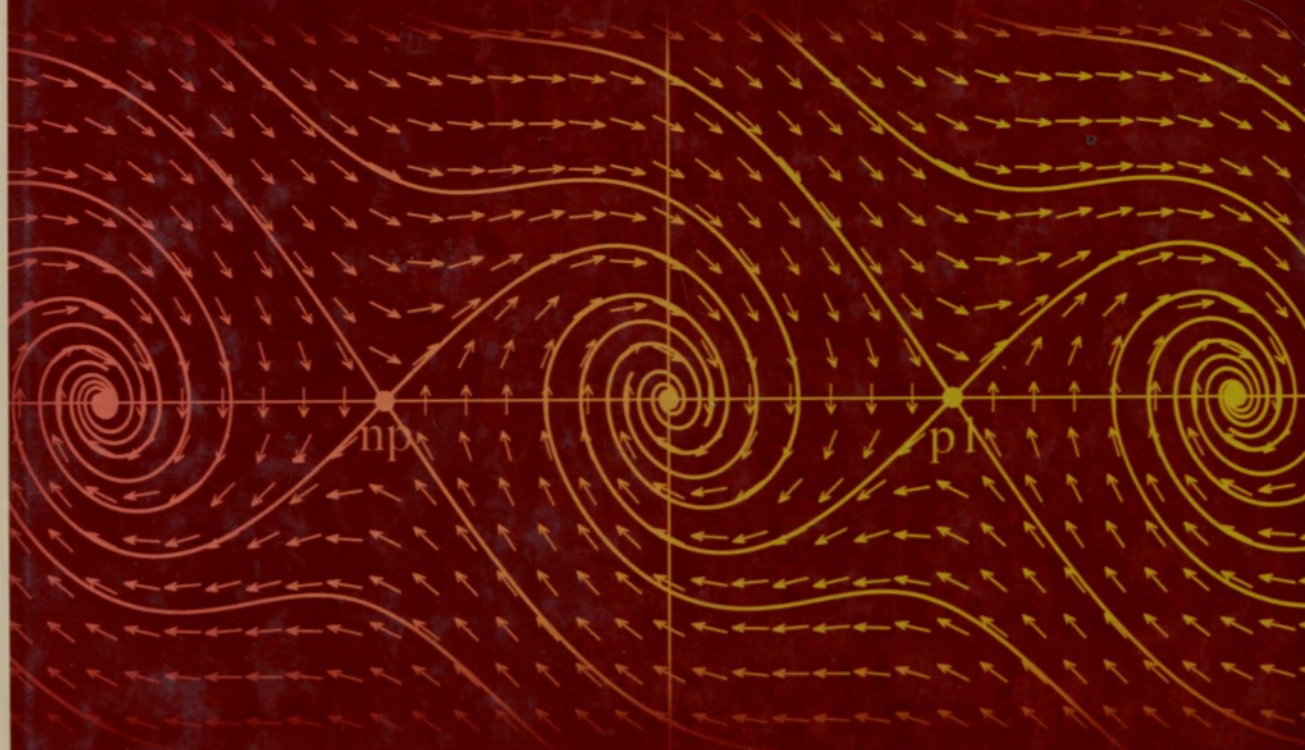


TEXTBOOKS IN MATHEMATICS

# ORDINARY DIFFERENTIAL EQUATIONS

An Introduction to the Fundamentals

SECOND EDITION



Kenneth B. Howell



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## About the author

**Kenneth B. Howell** earned bachelor's degrees in both mathematics and physics from Rose-Hulman Institute of Technology, and master's and doctoral degrees in mathematics from Indiana University. For more than thirty years, he was a professor in the Department of Mathematical Sciences of the University of Alabama in Huntsville. Dr. Howell published numerous research articles in applied and theoretical mathematics in prestigious journals, served as a consulting research scientist for various companies and federal agencies in the space and defense industries, and received awards from the college and university for outstanding teaching. He is also the author of *Fourier Analysis, Second Edition* (Chapman & Hall/CRC Press, 2016).



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