

Studies in Systems, Decision and Control 191

Ernesto Olguín Díaz

# 3D Motion of Rigid Bodies

A Foundation for Robot Dynamics  
Analysis

 Springer

# Contents

## Part I Fundamentals

<b>1 Mathematic Foundations</b> . . . . .	3
1.1 Matrices . . . . .	6
1.1.1 The Determinant of a Matrix . . . . .	10
1.1.2 Other Basic Functions of a Matrix . . . . .	16
1.1.3 Basic Matrix Operations . . . . .	17
1.1.4 Square Matrices . . . . .	31
1.1.5 Square Block Matrices . . . . .	38
1.2 Linear Operators . . . . .	43
1.2.1 Vector Spaces . . . . .	44
1.2.2 Transformations . . . . .	49
1.2.3 Range Space and Null Space . . . . .	56
1.2.4 Eigenvalues and Eigenvectors . . . . .	61
1.3 3D Vector Analysis . . . . .	77
1.3.1 The Dot (Scalar) Product . . . . .	80
1.3.2 The Cross (Vector) Product . . . . .	83
1.3.3 Vectorial Geometry . . . . .	88
1.3.4 Vector Differentiation . . . . .	91
<b>2 Classical Mechanics</b> . . . . .	101
2.1 Newton's Mechanics of a Particle . . . . .	101
2.1.1 The Work and Energy . . . . .	102
2.1.2 The Power . . . . .	107
2.1.3 Final Comments . . . . .	110
2.2 Lagrange Mechanics for Multi-particle Systems . . . . .	112
2.2.1 D'Alembert-Lagrange's Equation . . . . .	116
2.2.2 The Euler-Lagrange's Equation . . . . .	129
2.2.3 Properties of Lagrangian Formulation . . . . .	136
2.2.4 Final Comments . . . . .	143

## Part II Free Motion of Single Rigid Body

<b>3 Rigid Motion</b> . . . . .	147
3.1 Translations . . . . .	147
3.2 Rotations . . . . .	149
3.2.1 Attitude's Degrees of Freedom . . . . .	153
3.2.2 Basic Rotations . . . . .	154
3.2.3 Composed Rotations . . . . .	157
3.2.4 The Euler Theorem of Rotations . . . . .	160
3.3 The Rigid Motion Kinematics . . . . .	165
3.3.1 The Angular Velocity . . . . .	166
3.3.2 The Coriolis Effect . . . . .	179
3.3.3 The Linear Velocity and Acceleration . . . . .	180
3.3.4 Kinematic Equations . . . . .	182
<b>4 Attitude Representations</b> . . . . .	185
4.1 Pair of Rotation . . . . .	186
4.1.1 Computing the Axis/Angle Parameters from a Rotation Matrix . . . . .	188
4.1.2 The Angular Velocity for the Axis/Angle Representation . . . . .	189
4.1.3 Attitude Inverse Kinematics for the Axis/Angle Representation . . . . .	190
4.2 Roll-Pitch-Yaw Representation . . . . .	192
4.2.1 Computing the Roll, Pitch and Yaw Angles from a Rotation Matrix . . . . .	193
4.2.2 Angular Velocity and the Roll-Pitch-Yaw Representation . . . . .	195
4.2.3 Attitude Inverse Kinematics for the Roll-Pitch-Yaw Representation . . . . .	197
4.3 Euler Angles $zyz$ . . . . .	198
4.3.1 Computing the $zyz$ Euler Angles from a Rotation Matrix . . . . .	199
4.3.2 Angular Velocity and the $zyz$ Representation . . . . .	199
4.3.3 Attitude Inverse Kinematics for the Euler Angles $zyz$ Representation . . . . .	201
4.4 Unit Quaternions . . . . .	202
4.4.1 Equivalence with the Pair of Rotation . . . . .	204
4.4.2 Computing Quaternions from a Rotation Matrix . . . . .	208
4.4.3 Composed Rotation with Quaternions . . . . .	210
4.4.4 Angular Velocity with Quaternions . . . . .	212
4.4.5 Attitude Inverse Kinematics for the Quaternion Representation . . . . .	215



4.5	Rodrigues Parameters . . . . .	219
4.5.1	Computing Gibb's Vector from a Rotation Matrix . . . . .	220
4.5.2	Angular Velocity and Rodrigues Parameters . . . . .	221
4.5.3	Attitude Inverse Kinematics for Gibb's Representation . . . . .	222
4.6	Resume . . . . .	230
<b>5</b>	<b>Dynamics of a Rigid Body . . . . .</b>	<b>231</b>
5.1	The Center of Mass . . . . .	234
5.1.1	Kinematics at the Center of Mass . . . . .	238
5.2	The Kinetic Energy . . . . .	239
5.2.1	The Inertia Tensor . . . . .	241
5.2.2	Different Coordinates Expressions for the Kinetic Energy . . . . .	246
5.3	Momentums . . . . .	247
5.3.1	Linear Momentum Expression . . . . .	247
5.3.2	Angular Momentum Expression . . . . .	248
5.4	Rigid Dynamic Motion . . . . .	250
5.4.1	Euler's First Law of Motion . . . . .	251
5.4.2	Euler's Second Law of Motion . . . . .	252
5.4.3	The Newton–Euler Formulation . . . . .	255
5.5	Work and Power . . . . .	258
5.6	External Influences . . . . .	260
5.6.1	The Gravity Force . . . . .	261
5.7	Kirchhoff (Energy-Based) Formulation . . . . .	263
5.7.1	Kirchhoff–Euler Equivalence . . . . .	264
5.7.2	The Potential Energy . . . . .	266
<b>6</b>	<b>Spacial Vectors Approach . . . . .</b>	<b>273</b>
6.1	Spacial Vectors . . . . .	273
6.1.1	The Twist: the Velocity Spacial Vector . . . . .	273
6.1.2	The Wrench: the Force Spacial Vector . . . . .	274
6.1.3	The Pose . . . . .	275
6.1.4	The Motion and Force Spaces . . . . .	276
6.2	Spacial Vectors Transformations . . . . .	276
6.2.1	Extended Rotations . . . . .	276
6.2.2	Extended Translations . . . . .	278
6.2.3	The Spacial Vector Product . . . . .	281
6.3	Spacial Vector's Kinematics . . . . .	283
6.3.1	The Plücker Transformation . . . . .	283
6.3.2	The Pose Kinematics . . . . .	286
6.4	Spacial Vector's Dynamics . . . . .	288
6.4.1	The Power . . . . .	288
6.4.2	Kinetic Energy . . . . .	288

6.4.3	The Momentum's Spacial Vector . . . . .	290
6.4.4	Spacial Vector's Rigid Dynamic Motion . . . . .	292
6.4.5	Wrench of Exogenous Influences . . . . .	302
<b>7</b>	<b>Lagrangian Formulation . . . . .</b>	<b>307</b>
7.1	Direct Lagrangian Expression . . . . .	308
7.1.1	The Kinetic Energy and Inertia Matrix . . . . .	308
7.1.2	The Coriolis Matrix . . . . .	310
7.1.3	The Gravity Vector . . . . .	312
7.1.4	The Generalized Forces . . . . .	314
7.2	Indirect Lagrangian Formulation . . . . .	315
7.2.1	The quasi-Lagrangian Coordinates . . . . .	315
7.2.2	Indirect Lagrangian Equivalence . . . . .	316
7.2.3	Properties of Quasi-Lagrangian Formulation . . . . .	318
7.3	Conclusions . . . . .	326
<b>Part III Constraint Motion of a Single Rigid Body</b>		
<b>8</b>	<b>Model Reduction Under Motion Constraint . . . . .</b>	<b>331</b>
8.1	The Constraint Model . . . . .	332
8.2	Model Reduction, the Dynamical Approach . . . . .	334
8.2.1	Example 1: The Omnidirectional Mobile Robot . . . . .	335
8.2.2	Example 2: The Differential Mobile Robot . . . . .	346
8.3	Twist Coordinates Separation: The Kinematical Approach for the Dynamic Model Reduction . . . . .	351
8.3.1	Wrench Coordinates Separation . . . . .	354
8.3.2	Kinematical Reduction of the Dynamic Model . . . . .	355
8.3.3	Example 3: The Omnidirectional Mobile Robot, Kinematic Approach . . . . .	359
8.3.4	Example 4: The Differential Mobile Robot, Kinematic Approach . . . . .	364
8.4	Resume . . . . .	369
<b>Appendix A: The Cross Product Operator . . . . .</b>		<b>371</b>
<b>Appendix B: Fundamentals of Quaternion Theory . . . . .</b>		<b>385</b>
<b>Appendix C: Extended Operators . . . . .</b>		<b>401</b>
<b>Appendix D: Examples for the Center of Mass and Inertia Tensors of Basic Shapes . . . . .</b>		<b>411</b>

Ernesto Olguín Díaz

## 3D Motion of Rigid Bodies

A Foundation for Robot Dynamics Analysis

This book offers an excellent complementary text for an advanced course on the modelling and dynamic analysis of multi-body mechanical systems, and provides readers an in-depth understanding of the modelling and control of robots.

While the Lagrangian formulation is well suited to multi-body systems, its physical meaning becomes paradoxically complicated for single rigid bodies. Yet the most advanced numerical methods rely on the physics of these single rigid bodies, whose dynamic is then given among multiple formulations by the set of the Newton–Euler equations in any of their multiple expression forms.

This book presents a range of simple tools to express in succinct form the dynamic equation for the motion of a single rigid body, either free motion (6-dimension), such as that of any free space navigation robot or constrained motion (less than 6-dimension), such as that of ground or surface vehicles. In the process, the book also explains the equivalences of (and differences between) the different formulations.

Engineering

ISBN 978-3-030-04274-5

