

TEXTBOOKS IN MATHEMATICS

LINEAR ALGEBRA

AN INQUIRY-BASED APPROACH

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Linear Algebra: An Inquiry-Based Approach is written to give instructors a tool to teach students to develop a mathematical concept from first principles. The Inquiry-Based Approach is central to this development. The text is organized around and offers the standard topics expected in a first undergraduate course in linear algebra.

In our approach, students begin with a problem and develop the mathematics necessary to describe, solve, and generalize it. Thus, students learn a vital skill for the 21st century: the ability to create a solution to a problem.

This text is offered to foster an environment that supports the creative process. The twin goals of this textbook are:

- Providing opportunities to be creative
- Teaching “ways of thinking” that will make it easier to be creative

To motivate the development of the concepts and techniques of linear algebra, we include more than two hundred activities on a wide range of problems, from purely mathematical questions, through applications in biology, computer science, cryptography, and more.

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